‘Output Persistence from Monetary Shocks with Staggered Prices or Wages under a Taylor Rule’ by Sebastiano Daros and Neil Rankin

Technical Appendix

This Technical Appendix provides derivations of some key results in the paper which could not be included in the main paper for reasons of length.

3. Taylor-Style Staggering

Derivation of Determinacy Condition (3.8)

In the case of a linear differential equation of *n*-th order, the Routh-Hurwitz criterion can be applied to the corresponding characteristic polynomial $Q(s)$ in order to check the number n_c of its roots with positive real parts (the remaining n-n_c roots having negative real parts). In the case of a linear difference equation of *n*-th order, the number n_c of roots of its characteristic polynomial $P(z)$ that lie outside the unit circle (the remaining n-n_c roots lying inside) can be calculated by applying the Routh-Hurwitz criterion to the polynomial obtained by transforming $P(z)$ by means of the bilinear transformation $z=(1+s)/(1-s)$ (which is such that $\mathrm{Re}(s)<0 \Leftrightarrow |z|<1$). Due to space constraints, the steps of the Routh-Hurwitz test will not be presented (see DiStefano et al., 1990, ch. 5, for details).

i) The case $\gamma_w \neq \sigma^{-1}$

The (normalised) characteristic polynomial associated with equation (3.7)

$$P(z) \equiv z^3 + p_1 z^2 + p_2 z + p_3 = 0\,, \tag{T.1}$$

where $p_1 = \dfrac{k_{Tw} + (1+\beta)/\sigma - \beta\phi_\pi\gamma_w + \beta\phi_y}{\beta(\gamma_w - \sigma^{-1})}$, $p_2 = \dfrac{\gamma_w - \sigma^{-1} - k_{Tw}\phi_\pi - \phi_y}{\beta(\gamma_w - \sigma^{-1})}$, $p_3 = -\dfrac{\phi_\pi\gamma_w}{\beta(\gamma_w - \sigma^{-1})}$,

must have two roots outside the unit circle and one inside, in order to guarantee determinacy. By applying the bilinear transformation, $P(z)=0$ can be rewritten as

$$Q(s) \equiv s^3 + a_1 s^2 + a_2 s + a_3 = 0\,, \tag{T.2}$$

where

$$\begin{aligned}
a_1 &= \frac{3-p_1-p_2+3p_3}{1-p_1+p_2-p_3} = -\frac{\left(k_{Tw}+(1+\beta)\gamma_w\right)(\phi_\pi-1)+4(\beta-\phi_\pi)\gamma_w-4\beta\sigma^{-1}+\phi_y(1-\beta)}{\left(2\sigma^{-1}+\phi_y\right)(1+\beta)+\left(k_{Tw}-(1+\beta)\gamma_w\right)(1+\phi_\pi)} \\
a_2 &= \frac{3+p_1-p_2-3p_3}{1-p_1+p_2-p_3} = -\frac{\left(k_{Tw}-(1+\beta)\gamma_w\right)(1+\phi_\pi)+4(\beta+\phi_\pi)\gamma_w+2\sigma^{-1}(1-\beta)+\phi_y(1+\beta)}{\left(2\sigma^{-1}+\phi_y\right)(1+\beta)+\left(k_{Tw}-(1+\beta)\gamma_w\right)(1+\phi_\pi)} < 0 \cdot \\
a_3 &= \frac{1+p_1+p_2+p_3}{1-p_1+p_2-p_3} = \frac{\left(k_{Tw}+(1+\beta)\gamma_w\right)(\phi_\pi-1)+(1-\beta)\phi_y}{\left(2\sigma^{-1}+\phi_y\right)(1+\beta)+\left(k_{Tw}-(1+\beta)\gamma_w\right)(1+\phi_\pi)}
\end{aligned}$$

Note that $k_{Tw} \geq (1+\beta)\gamma_w$ by definition (see equation (3.5)).

The Routh-Hurwitz criterion implies that $Q(s)=0$ has two roots with positive real part and one root with negative real part (i.e. $P(z)=0$ has two roots outside the unit circle and one inside) if and only if (iff)

$$a_3 > 0 \ \text{ AND } \ \neg\left\{a_1 > 0 \ \text{ AND } \ a_2 - \frac{a_3}{a_1} > 0\right\}, \tag{T.3}$$

(3.8) is the necessary and sufficient condition for (T.3) to hold since

$$\begin{aligned}
&a_3 > 0 \ \Leftrightarrow \ \phi_\pi + \frac{1-\beta}{k_{Tw}\left[1+(1+\beta)\dfrac{\gamma_w}{k_{Tw}}\right]}\phi_y > 1 \\
&\{a_3 > 0 \ \text{ AND } \ a_2 < 0\} \ \Rightarrow \ \neg\left\{a_1 > 0 \ \text{ AND } \ a_2 - \frac{a_3}{a_1} > 0\right\}
\end{aligned} \quad . \tag{T.4}$$

ii) The case $\gamma_w = \sigma^{-1}$

The characteristic polynomial associated with equation (3.7)

$$P(z) \equiv p_0 z^2 + p_1 z + p_2 = 0, \tag{T.5}$$

where $p_0 = k_{Tw} + (1+\beta)\gamma_w - \beta\phi_\pi\gamma_w + \beta\phi_y$, $p_1 = -\left(k_{Tw}\phi_\pi + \phi_y\right) < 0$, $p_2 = -\phi_\pi\gamma_w < 0$, must have one root outside the unit circle and one inside, in order to guarantee determinacy. By applying the bilinear transformation, $P(z)=0$ can be rewritten as

$$Q(s) \equiv a_0 s^2 + a_1 s + a_2 = 0, \tag{T.6}$$

where

$$\begin{aligned}
a_0 &= p_0 - p_1 + p_2 = \left(k_{Tw}-(1+\beta)\gamma_w\right)(1+\phi_\pi) + 2(1+\beta)\gamma_w + (1+\beta)\phi_y > 0 \\
a_1 &= 2(p_0 - p_2) = 2\left(k_{Tw} + (1+\beta)\gamma_w + (1-\beta)\phi_\pi\gamma_w + \beta\phi_y\right) > 0 \\
a_2 &= p_0 + p_1 + p_2 = -\left[\left(k_{Tw}+(1+\beta)\gamma_w\right)(\phi_\pi - 1) + (1-\beta)\phi_y\right]
\end{aligned} \quad .$$

The Routh-Hurwitz criterion implies that $P(s)$ has one root with positive real part and one root with negative real part iff

$$\frac{a_2}{a_0} < 0 . \qquad \text{(T.7)}$$

In passing, note that the actual sign of a_1 is irrelevant for determinacy. Since $a_0 > 0$,

$$\frac{a_2}{a_0} < 0 \iff a_2 < 0 \iff \phi_\pi + \frac{1-\beta}{k_{Tw}\left[1+(1+\beta)\frac{\gamma_w}{k_{Tw}}\right]}\phi_y > 1 . \qquad \text{(T.8)}$$

QED.

Derivation of Persistence Condition (3.9)

i) The case $\gamma_w \neq \sigma^{-1}$

Under the determinacy condition (3.8), the (normalised) characteristic polynomial (T.1) has one and only one root λ_S inside the unit circle. Therefore, by continuity,

$$\begin{aligned} &\lambda_S = 0 \iff P(z=0) = p_3 = -\frac{\phi_\pi \gamma_w}{\beta\left(\gamma_w - \sigma^{-1}\right)} = 0 \iff \phi_\pi \gamma_w = 0 \\ &0 < \lambda_S < 1 \iff \frac{P(z=0)}{P(z=1)} = \frac{p_3}{1+p_1+p_2+p_3} = \frac{\phi_\pi \gamma_w}{\left(k_{Tw}+(1+\beta)\gamma_w\right)(\phi_\pi - 1) + (1-\beta)\phi_y} < 0 \end{aligned} \qquad \text{(T.9)}$$

The proof is concluded by noting that, if neither of the conditions (T.9) hold, it must be true that $-1 < \lambda_S < 0$.

ii) The case $\gamma_w = \sigma^{-1}$

In this case, from the characteristic polynomial (T.5) it follows that

$$\begin{aligned} &\lambda_S = 0 \iff P(z=0) = p_2 = -\phi_\pi \gamma_w = 0 \\ &0 < \lambda_S < 1 \iff \frac{P(z=0)}{P(z=1)} = \frac{p_2}{p_0+p_1+p_2} = \frac{\phi_\pi \gamma_w}{\left(k_{Tw}+(1+\beta)\gamma_w\right)(\phi_\pi - 1) + (1-\beta)\phi_y} < 0 \end{aligned} \qquad \text{(T.10)}$$

The proof is concluded by noting that, if neither of the conditions (T.10) hold, it must be true that $-1 < \lambda_S < 0$.

iii) The case $\phi_\pi < 0$

It is clear from conditions (T.9) and (T.10) that the model could generate persistence under the alternative assumption that ϕ_π were negative. (Remember that in the Taylor Rule (3.6) the output and inflation response coefficients was assumed positive). However, this is true only if the equilibrium is still uniquely determined. In fact, it is easy to see from the proof of the determinacy condition (3.8) above, that determinacy still holds for moderately negative ϕ_π. However, in this case, (3.8) is not in general a sufficient condition for determinacy, but only necessary. In passing, note that, if $\gamma_w \neq \sigma^{-1}$, (3.8) and $\phi_\pi > -\beta$ would be sufficient conditions for determinacy (since a_2 in (T.3) is still negative). If $\gamma_w = \sigma^{-1}$, on the other hand, (3.8) and $\phi_\pi > -1$ would be sufficient conditions (since a_0 in (T.7) is still positive).

iv) The case $\phi_\pi = 0$

It is clear from (3.7) that a Taylor Rule that does not feed back on inflation causes y_{t-1} to drop out. Therefore in this case the model is completely forward-looking, which implies that it cannot generate output persistence at all (and the above analysis of the determinacy condition no longer applies).

4. Calvo-Style Staggering

Derivation of Persistence Condition (4.7)

Under the determinacy condition (4.6), the (normalised) characteristic polynomial $P(z)$ associated with equation (4.5)

$$P(z) \equiv z^3 + p_1 z^2 + p_2 z + p_3 = 0, \tag{T.11}$$

where $p_1 = -\dfrac{k_{Cw} + (1+\beta)/\sigma + \beta\phi_\pi\eta_{mpl,Y} + \beta\phi_y}{\beta\left(\eta_{mpl,Y} + \sigma^{-1}\right)} < 0$, $p_2 = \dfrac{\eta_{mpl,Y} + \sigma^{-1} + k_{Cw}\phi_\pi + \phi_y}{\beta\left(\eta_{mpl,Y} + \sigma^{-1}\right)} > 0$,

$p_3 = -\dfrac{\phi_\pi\eta_{mpl,Y}}{\beta\left(\eta_{mpl,Y} + \sigma^{-1}\right)} < 0$, has one and only one root λ_S inside the unit circle. Therefore, by continuity,

$$\begin{aligned} &\lambda_S = 0 \iff P(z=0) = p_3 = -\frac{\phi_\pi\eta_{mpl,Y}}{\beta\left(\eta_{mpl,Y} + \sigma^{-1}\right)} = 0 \quad \text{iff} \quad \phi_\pi\eta_{mpl,Y} = 0 \\ &0 < \lambda_S < 1 \iff \frac{P(z=0)}{P(z=1)} = \frac{p_3}{1 + p_1 + p_2 + p_3} = -\frac{\phi_\pi\eta_{mpl,Y}}{\left(k_{Cw} - (1+\beta)\eta_{mpl,Y}\right)(\phi_\pi - 1) + (1-\beta)\phi_y} < 0 \end{aligned} \tag{T.12}$$

The proof is concluded by noting that, if neither of the conditions (T.12) hold, it must be true that $-1 < \lambda_S < 0$.

References

DiStefano, J. J., Stubberud, A. R., & Williams, I. J., 1990. Schaum's Outline of Theory and Problems of Feedback and Control Systems (2nd ed.). New York, London: McGraw-Hill.